Adaptive Newton Methods For Semilinear Problems With Singular Perturbations

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Outline

Problem formulation

A Posteriori Error Analysis

An Adaptive Newton-Galerkin Algorithm

Numerical Examples

Semilinear Elliptic Problems

- Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2\}$, and $\varepsilon > 0$. Find $u : \Omega \to \mathbb{R}$, s.t.

$$\begin{cases} -\varepsilon \Delta u = f(u), & \Omega, \\ u = 0, & \partial\Omega. \end{cases} \tag{1}$$

- $f : \mathbb{R} \to \mathbb{R}$ sufficiently smooth nonlinearity.
- Possibly with $\varepsilon \ll 1$.

Example

Ginzburg-Landau Equation:

$$\begin{cases} \varepsilon \Delta u = u^3 - u, & \Omega, \\ u = 0, & \partial\Omega. \end{cases}$$

Possible challenges:

- Solutions of (1) are typically not unique and may
- exhibit boundary layers, interior shocks and (multiple) spikes.

Problem Formulation

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- $X = H_0^1(\Omega)$
- Define $\mathsf{F}_\varepsilon : X \to X'$ through

$$\langle \mathsf{F}_\varepsilon(u), v \rangle := \int_\Omega \{\varepsilon \nabla u \cdot \nabla v - f(u)v\} \, \mathrm{dx}.$$

- Then any solution $u \in X$ of (1) solves

$$\mathsf{F}_\varepsilon(u) = 0, \quad \text{in } X'.$$

Newton Linearization

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- Given $u_0 \in X$.
- Find $u_{n+1} \in X$ from
- $u_n \in X, \Delta t_n \in (0,1], n \geq 0$, s.t.

$$\mathsf{F}'_\varepsilon(u_n)(u_{n+1} - u_n) = -\Delta t_n \mathsf{F}_\varepsilon(u_n), \quad \text{in } X',$$

where Δt_n is chosen adaptively.

Well-Posedness

- Given $\underline{\lambda}, \overline{\lambda} \geq 0$ with

$$-\underline{\lambda} \leq f'(u) \leq \overline{\lambda}, \quad u \in \mathbb{R}.$$

- Suppose

$$\varepsilon C_P^{-2} > \overline{\lambda}, \quad (C_P \text{ the Poincaré constant on } \Omega.).$$

Then for any given $u_n \in X$ there is an unique $u_{n+1} \in X$ s.t.

$$\mathsf{F}_\varepsilon'(u_n)(u_{n+1} - u_n) = -\Delta t_n \mathsf{F}_\varepsilon(u_n) \quad \text{in} \quad X'.$$

Linear FEM Discretization

On a partition $\mathcal{T}_h$ of Ω define the finite element space

$$V_0^h := \{\varphi \in H_0^1(\Omega) : \varphi|_T \in \mathbb{P}_1(T) \,\forall T \in \mathcal{T}_h\}.$$

Find $u_{n+1}^h \in V_0^h$ from $u_n^h \in V_0^h$, s.t.

$$\left\langle \mathsf{F}_\varepsilon'(u_n^h)(u_{n+1}^h - u_n^h), v \right\rangle = \left\langle -\Delta t_n \mathsf{F}_\varepsilon(u_n^h), v \right\rangle, \quad v \in V_0^h.$$

Linear FEM Discretization

$$\left\langle \mathrm{F}_\varepsilon'(u_n^h)(u_{n+1}^h - u_n^h), v \right\rangle = \left\langle -\Delta t \mathrm{F}_\varepsilon(u_n^h), v \right\rangle, \qquad v \in V_0^h,$$

$$\Leftrightarrow \qquad \varepsilon \int_\Omega \nabla u_{n+1}^{(\Delta t,h)} \cdot \nabla v \mathrm{dx} = \int_\Omega f^{\Delta t}(u_{n+1}^h) v \mathrm{dx}, \qquad v \in V_0^h,$$

with

$$u_{n+1}^{(\Delta t,h)} := u_{n+1}^h - (1 - \Delta t) u_n^h,$$

$$f^{\Delta t}(u_{n+1}^h) := \Delta t \; f(u_n^h) + f'(u_n^h)(u_{n+1}^h - u_n^h).$$

Notation/Robustness

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Equip X with the norm

$$\|u\|_{\varepsilon,D} := \left(\varepsilon\|\nabla u\|_{0,D}^2 + \|u\|_{0,D}^2\right)^{1/2}, \ D \subset \Omega.$$

Thus

$$\|\varphi\|_{X',\varepsilon} = \sup_{x\in X\setminus 0} \frac{\langle\varphi, x\rangle}{\|x\|_\varepsilon}.$$

Following along the lines of [Verfürth; 1996], we set for

A Posteriori Analysis/Upper Bound

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$T \in \mathcal{T}_h, \alpha_T := \min(1, \varepsilon^{-1/2} h_T), \alpha_E := \min(1, \varepsilon^{-1/2} h_E)$, and define

$$\delta_{n,T} := \left\| f^{\Delta t}(u^h_{n+1}) - f(u^{(\Delta t,h)}_{n+1}) \right\|_{0,T},$$

$$\eta^2_{n,T} := \alpha_T^2 \left\| f^{\Delta t}(u^h_{n+1}) + \varepsilon \Delta u^{(\Delta t,h)}_{n+1} \right\|^2_{0,T} + \frac{1}{2} \sum_{E \in \mathcal{E}_h(T)} \varepsilon^{-1/2} \alpha_E \left\| \varepsilon \left[\!\left[\nabla u^{(\Delta t,h)}_{n+1} \right]\!\right] \right\|^2_{0,E}.$$

Theorem

There holds the upper a posteriori error bound:

$$\left|\!\left|\!\left| \mathsf{F}_\varepsilon(u^{(\Delta t,h)}_{n+1}) \right|\!\right|\!\right|^2_{X',\varepsilon} \preccurlyeq \delta^2_{n,\Omega} + \sum_{T \in \mathcal{T}_h} \eta^2_{n,T}. \tag{2}$$

An Adaptive Algorithm

Algorithm

Given a parameter $\theta > 0$*, a (coarse) starting mesh* $\mathcal{T}_h$ *in* Ω*, and an initial guess* $u_0^h \in V_0^h$*. Set* $n := 0$*.*

1. *Determine a Newton step size parameter* Δt_n *based on* u_n^h *by an adaptive procedures.*
2. *Compute the FEM solution* u_{n+1}^h *with step size* Δt_n *on the mesh* $\mathcal{T}_h$ *and evaluate the error indicators* $\eta_{T,n}$*,* $T \in \mathcal{T}_h$*,* $\delta_{n,\Omega}$*.*
3. *If*

$$\delta_{n,\Omega}^2 \leq \theta \sum_{T \in \mathcal{T}_h} \eta_{T,n}^2, \tag{3}$$

   refine the mesh $T \in \mathcal{T}_h$ *using the indicators* $\eta_{n,T}$*,* $T \in \mathcal{T}_h$*; repeat step (2) with the new mesh* $\mathcal{T}_h$*; Otherwise; set* $n \leftarrow n+1$*, and perform another adaptive Newton step by going back to (1).*

An Adaptive Algorithm

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- Basic idea of fully adaptive Newton Galerkin method:
- Provide an interplay between an adaptive Newton iteration and a robust adaptive FEM.
  See also [El-Alaoui, Ern, Vohralík; 2011].

An Adaptive Algorithm

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Remark

- *Whenever the mesh will be refined, u_n^h is interpolated on the refined mesh and defines the new initial guess.*
- *The linear systems resulting from the finite element discretization are solved by means of a direct solver; in this way, this approach differs from inexact Newton methods.*
  See also [Ern, Vohralík, 2013].

Linear Singular Perturbed Problem

We test robustness using the linear problem:

$$\begin{cases} -\varepsilon u'' = 1 - u, & \Omega, \\ u = 0, & \partial\Omega. \end{cases}$$

Linear Singular Perturbed Problem

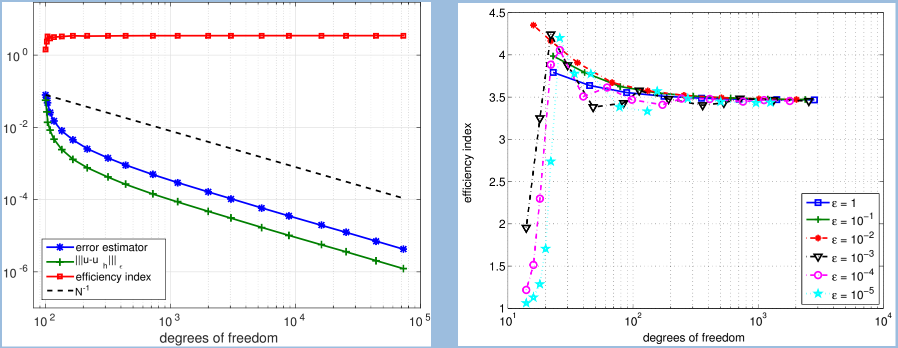

Figure: Performance for $\varepsilon = 10^{-5}$ (left) and efficency indices (right).

Fisher's Equation

$$\begin{cases} \varepsilon u'' = u^2 - u, \quad (0,1), \\ u(0) = \alpha, \\ u(1) = \beta. \end{cases}$$

For $\alpha > -1/2$ and $\beta < 1$, the solutions feature an increasing number of spikes (which are bounded by 1).

Fisher's Equation

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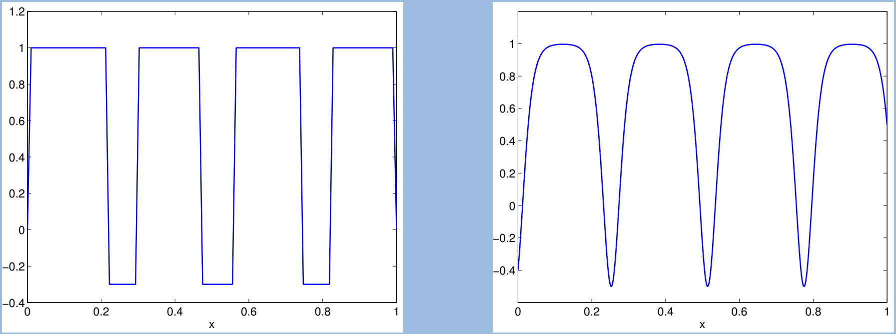

Figure: Initial data (left) and numerical solution resulting from Algorithm 1 (right) with $\alpha = -0.4$, $\beta = 0.5$, and $\varepsilon = 0.00025$.

Fisher's Equation

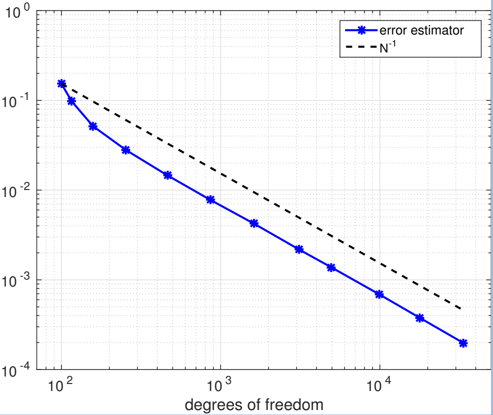

Figure: Estimated error for $\varepsilon = 0.00025$.

Ginzburg/Landau Equation

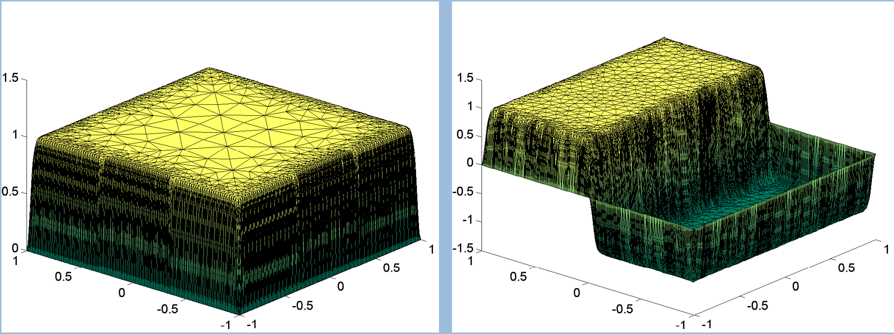

Figure: Numerical solutions with $\varepsilon = 0.00025$.

Ginzburg/Landau Equation

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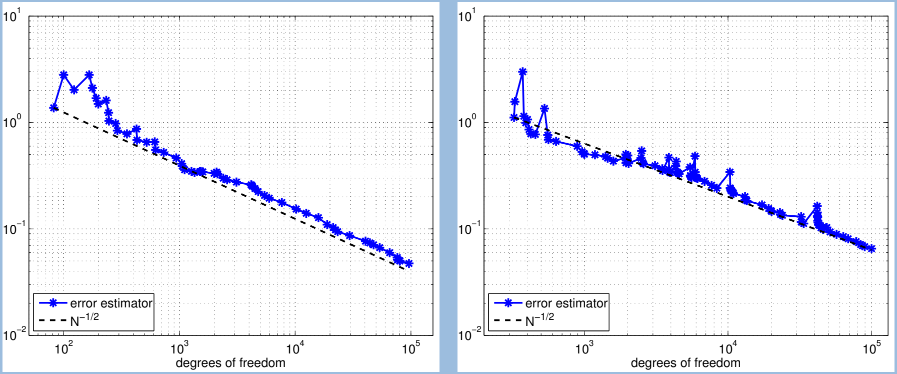

Figure: Performance with $\varepsilon = 0.00025$.

Summary

- Combine an adaptive Newton step size method with an
- automatic mesh refinement linear FEM procedure.
- Furthermore, the sequence of linear problems is treated by means of a robust error analysis (with respect to the singular perturbations),
- to robustly resolve the singular perturbations at an optimal rate.

References

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